<table>
<tr><th colspan="3">TOPIC PLAN</th></tr>
<tr><td>Partner organization</td><td colspan="2">Goce Delcev University – Stip, North Macedonia</td></tr>
<tr><td>Topic</td><td colspan="2">Double Integrals: Calculating Area</td></tr>
<tr><td>Lesson title</td><td colspan="2">Calculating Area Using Polar Coordinates</td></tr>
<tr><td>Learning objectives</td><td>✓ Students will acquire and deal with double integrals;
✓ Students will be able to estimate area of different 2D shapes, including shapes which border is constructed with circles;
✓ Students will be able to deal with different problems in everyday life, which require calculating area;
✓ Students are encouraged to use technology and different software in their work, while considering problem - based situations.</td><td rowspan="2">Strategies/Activities
☐Graphic Organizer
☐Think/Pair/Share
☐Modeling
☐Collaborative learning
☐Discussion questions
☐Project based learning
☐Problem based learning

Assessment for learning
☐Observations
☐Conversations
☐Work sample
☐Conference
☐Check list
☐Diagnostics

Assessment as learning
☐Self-assessment
☐Peer-assessment
☐Presentation
☐Graphic Organizer</td></tr>
<tr><td>Aim of the lecture / Description of the practical problem</td><td>The aim of the lecture is to make students able to calculate area of 2D - shapes which border is constructed with circles. It is easier to be done using polar coordinates.

The teacher gives the next problem to the students:

A children's playground has the shape of two circles with equal radius R=8 meters and central distance d=8 meters. In the intersection of the two circles, there are children's requisites for playing, and outside the intersection, on both sides, there is a green-grass area for playing. In order to maintain the playground, it is necessary to know the area of the green-grass section and the area of the section with children's requisites. Try to calculate both of them.

The teacher encourages students to work on the problem and collaborate in order to find a solution of the problem. The teacher encourages students to use appropriate software to plot the circles described in the problem.</td></tr>
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<td>Previous knowledge assumed:</td>
<td>
- analytic geometry, sketching figures in Dekart coordinate system
- differentiation of functions with two real variables
- algebraic equations
- calculating double integrals
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<td rowspan="2">☐Homework

Assessment of learning
☐Test
☐Quiz
☐Presentation
☐Project
☐Published work</td>
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<td>Introduction / Theoretical basics</td>
<td>With the definition of the double integral, students are introduced with its application in calculating area, i.e. the value of the integral $\iint_D dxdy$ is equal to the area of the region $D \subseteq \mathbf{R}^2$. When D is circle or another shape constructed with circles, the integral above is easier to be calculated using polar coordinates, i.e substituting $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$. If the pair (x, y) represents coordinates of a point within the circle $x^2 + y^2 = R^2$, such that the distance between the point and $(0,0)$ is ρ and the angle between that segment with length ρ and positive way of x-axes is φ, then according to the definition of trigonometric functions, $(x, y) = (\rho\cos\theta, \rho\sin\theta)$. For the points within the circle, the distance to $(0,0)$ can be maximum R, thus $0 \le \rho \le R$. For the points within the circle for the angle φ holds: $0 \le \varphi \le 2\pi$. If we have to calculate the integral $\iint_D dxdy$ over the region D: $x^2 + y^2 = R^2$, we have to calculate $\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy$. If we make substitution with polar coordinates, we have to calculate the integral with rectangular region of integration: $\int_0^{2\pi} d\varphi \int_0^R \rho d\rho$ which is far easier and faster to calculate then the previous one. That is the advantage of</td>
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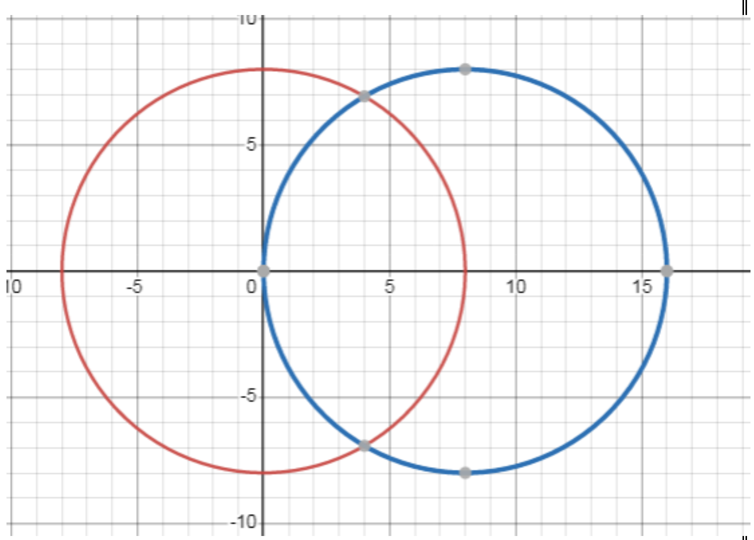
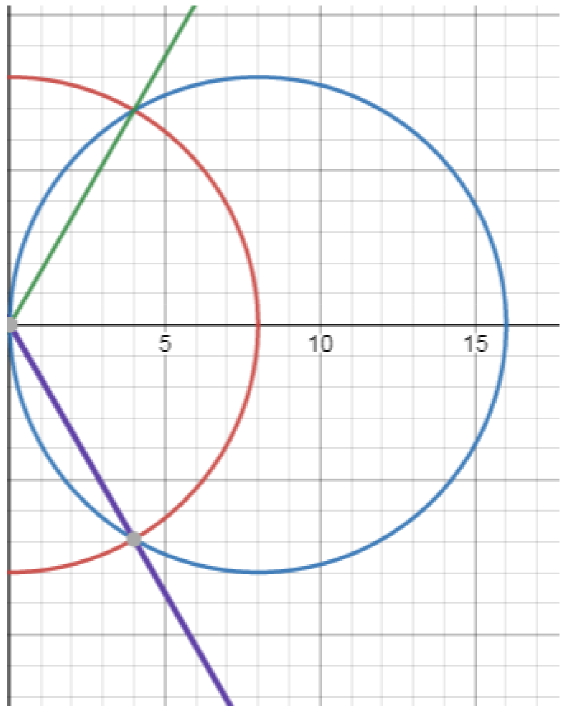
	introducing polar coordinates. Teacher can solve students some examples using polar coordinates before solving the main problem set at the beginning of the lesson. Different software can be used to plot the lines that determine the region in the examples.	
Action	To solve the given problem, we can plot the circles that form the children's playground such that coordinates beginning $(0,0)$ is in the center of one of the circles and the segment between the centers of the circles lies on the x-axes. The equation for the first circle will be $x^2+y^2=64$ and for the second one it will be $(x-8)^2+y^2=64$.  If we write x and y in polar form, $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$ and substitute it in the equations of both circles, we can conclude that that the distance ρ between each point within the second circle that is right of the intersection, satisfies $8\le\rho\le16\cos\varphi$. To determine the angle φ for those points, we determine first the intersection points of the two circles and plot	

lines between each intersection point and coordinates beginning. One of the lines of $y = x\sqrt{3}$ and the other one is $y = -x\sqrt{3}$.



The angle between the line above the x-axes and x-axes is $\pi/3$ and the angle between the line bellow the x-axes and x-axes is $-\pi/3$. Thus, $-\pi/3 \le \varphi \le \pi/3$. Then the area within the second circle and right of the intersection is equal with the integral $\int\limits_{-\pi/3}^{\pi/3} d\varphi \int\limits_{8}^{16\cos\varphi} \rho d\rho$.

As in the children's playground there is the same area left of the intersection, thus the area with the green grass will be $P = 2\int\limits_{-\pi/3}^{\pi/3} d\varphi \int\limits_{8}^{16\cos\varphi} \rho d\rho$.

The area of the intersection (the area with the playing requisites in the playground) can be calculated as the difference of the circle area and the area left of the intersection, i.e. $P_1 = 64\pi - \int\limits_{-\pi/3}^{\pi/3} d\varphi \int\limits_{8}^{16\cos\varphi} \rho d\rho$.

Materials / equipment / digital tools / software	Literature given in the references at the end of the document / Digital device which supports software for plotting graph or can be used to plot online Various online graph plotter or software for graph plotting (Geogebra or similar)	
Consolidation	With the given examples students can consider that double integrals are important for solving real life problems. Students will learn how to get calculating double integrals easier, with introducing polar coordinates. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.	

Reflections and next steps	
Activities that worked	**Parts to be revisited**
Problem solving, collaboration, using technology	Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.

References

[1] E. Atanasova, S. Georgieva (2002), *Matematika 2*, Universitet “Sv. Kiril I Metodij” - Skopje
[2] S. Calaway D. Hoffman and D.Lippman (2014) “Applied Calculus”
[3] *P.D. Lax*, M. S.Terrell (2014) “Calculus with Applications”, Springer